When $\delta > \delta_2$, a unique boundary of the region of the cycle stability remains (Fig. 6, curve 5; $\delta = 3.161$) with a break when $K_0 = K_m$. In the limit when $\delta \gg 1$, we use expansions in terms of the parameter δ^{-2} to obtain the expression for the increment

$$\lambda_{\pm} = -(\gamma - K_0^2 + K^2) \pm [(\gamma - K_0^2)^2 + 4K_0^2K^2]^{1/2}$$

and this implies that the boundary of the region of stability of an oscillatory region has the form $\gamma = 3K_0^2$ when $K_0^2 > \delta^2/2$. When $K_0^2 < \delta^2/2$, the boundary coincides with the boundary of existence of the cycles $\gamma = \delta^2 + K_0^2$.

REFERENCES

1. VOZOVOI L.P. and NEPOMNYASHCHII A.A., Convection in a horizontal layer with spatial temperature modulation at the boundaries. In book: Hydrodynamics. Ed. 7, Perm', Izd-e Permsk. gos. ped. in-ta, 1974.
2. KELLY R.E. and PAL D., Thermal convection with spatiallv periodic boundary conditions: resonant wavelength excitation. J. Fluid Mech., Vol. 86, pt. 3, 1978.
3. BATISHCHEV V.A., KOLESOV V.V., SLITINSKAYA S.K. and YODOVICH V.I., Effect of spatial modulation of the temperature field on the stability of two-dimensional stationary flow in a horizontal layer of fluid. Izv. Akad.Nauk SSSR, MZhG, No. 3, 1983.
4. VOZOVOI L.P., Convection in a vertical layer with wave-like boundaries. Izv. Akad. Nauk SSSR, MZhG, No. 2, 1976.
5. VOZOVOI L.P. and NEPOMNYASHCHII A.A., On the stability of spatially periodic convective flows in a vertical layer with wave-like boundaries. PMM Vol. 43, No. 6, 1979.
6. LEVINA G.V. and NEPOMNYASHCHII A.A., On mixed convection modes in a vertical layer with unsteadily deformable boundaries. PMM Vol. 47, No. 3, 1983.
7. GERSHUNI G.Z. and ZHUKOVITSKII E.M., Convective stability of incompressible fluid. Moscow, Nauka, 1972.
8. LOBOV N.I., On the stability of mixed convective flow in a vertical plane layer. Izv. Akad. Nauk SSSR, MZhG, No. 6, 1979.
9. STEVARTSON K., STUART J.T., A non-linear instability theory for a wave system in plane Poiseuille flow. J. Fluid Mech., Vol. 48, pt. 3, 1971.

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WEAKLY SUPERCRITICAL DISSIPATIVE STRUCTURES ON CURVED SURFACES *

B.A. MALOMED and I.E. STAROSEL'SKII

Cellular, low amplitude structures appearing at cylindrical and spherical fronts of gaseous combustion and laser evaporation are described. In the case of a spherical front all these structures are found to be unstable. When the cylindrical front of gaseous combustion is expanded, we must expect the quasi one-dimensional structure homogeneous with respect to the ignorable coordinate to be replaced by a parquet-like pattern of rectangular cells, and later to reach a non-stationary regime. On the cylindrical front of laser evaporation the quasi one-dimensional structure of maximum amplitude is globally stable.

The best known hydrodynamic example of a kinetic problem connected with the formation of dissipative structures i.e. thermodynamically non-equilibrium stationary structures appearing as a result of the development of aperiodic instability in a spatially homogeneous state, are Benard cells /1,2/. New problems of this kind are connected with the instability of plane fronts of laser evaporation of condensed material, and of gaseous combustion /3—5/. The instability is aperiodic and appears at finite values of the wave number of the perturbation representing curvature of a plane front. The development of the instability leads to the formation of a stationary, periodically curved front /3/.

The purpose of this paper is to investigate such structures and their stability on cylindrical and spherical surfaces, and this corresponds to

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the problem of the propagation of a cylindrical or spherical flame through a gas, and of the laser evaporation of a spherical sample. Problems dealing with dissipative structures on curved surfaces are also of interest in biophysics, where a spherical surface models a cell membrane, while the cylindrical surface models the axon /6/.

1. In the weakly supercritical case, i.e. immediately after the loss of stability in the unperturbed solution, the evolution of the gas flame shape is described by the following dimensionless equation /7/:

$$\xi_t + \xi + 2\alpha\Delta\xi + \Delta^2\xi + (\nabla\xi)^2 = 0, \quad \alpha = \sqrt{R/R_0} \tag{1.1}$$

Here $\xi(t, x, y)$ is the coordinate of an individual point of the flame in the local associated reference system in which the unperturbed cvlindrical flame is described by the trivial solution $\xi \equiv 0$; the gradient ∇ and the Laplacian Δ act on the x, y coordinates directed along the unperturbed flame, R is the radius of curvature of the flame and R_0 is the value of the radius at which the trivial solution begins to lose its stability. The dimensionless coordinates ξ, x, y are measured in units of thermal thickness of the flame l, and the time is measured in units of l/U, where U is the normal flame propagation velocity. Equation (1.1) was derived from the Navier-Stokes, thermal conduction and diffusion equations on the assumption that the combustion activation energy is much higher than the termperature at the front, the diffusion coefficient and thermal conductivity are nearly equal, and the gas density change at the front is small.

Evaporation of condensed matter in the laser radiation field is described by the following phenomenological equation /8/ .

$$\xi_t + \xi + 2\alpha\Delta\xi + \Delta^2\xi + \xi^3 = 0 \tag{1.2}$$

Here α is a parameter proportional to the radiation intensity, and the remaining notation follows that of (1.1).

2. In the case of a cylindrical surface we use y as the cyclic coordinate $(0 \leqslant y < 2\pi R)$, and direct x along the generatrix of the cylinder. Then the wave vector $\mathbf{k}$ of a small perturbation can have an arbitrary component k_x, and its k_y component is "quantized", i.e.

$$k_y = nR^{-1}, \quad n = 0, 1, 2 \ldots \tag{2.1}$$

When $\alpha > 1$, the trivial solution $\xi \equiv 0$ loses its stability with respect to the perturbations with wave vectors belonging to any of the following domains:

$$\varepsilon^2 (k_x^2 + n^2R^{-2}) \geqslant 0, \; k_y = nR_0^{-1}, \; n \leqslant [R_0] \tag{2.2}$$

$$\varepsilon^2 (k^2) \equiv 2(\alpha - 1) - (k^2 - 1)^2 \tag{2.3}$$

We see that the condition of weak supercriticality implies that the quantity $\lambda^2 \equiv 2(\alpha - 1)$ and hence ε^2, are small. In the case of (1.2) we have various, locally stable solutions on the cylindrical surface. The simplest case represents the quasi one-dimensional structure /8/

$$\xi(x) = \sqrt{4/3}\,\varepsilon(k^2)\cos kx + O(\varepsilon^3) \tag{2.4}$$

where k belongs to the domain (2.2) when $n = 0$. We also have a solution in the form of a lattice of rectangular cells

$$\xi = {}^2/_3\,[(2\varepsilon^2(k_y^2) - \varepsilon^2(k_x^2))^{1/2}\cos kx + (2\varepsilon^2(k_x^2) - \varepsilon^2(k_y^2))^{1/2}\cos ky] \tag{2.5}$$

where the vector (k_x, k_y) belongs to one of the domains (2.2) when $n \neq 0$, as well as two solutions in the form of triangular cell lattices

$$\xi = \sqrt{4/15}\,\varepsilon(k^2)\left[\sin kx + \sin\frac{k}{2}(\sqrt{3}y - x) - \sin\frac{k}{2}(\sqrt{3}y + x)\right] \tag{2.6}$$

$$\xi = \sqrt{4/15}\,\varepsilon(k^2)\left[\sin ky + \sin\frac{k}{2}(\sqrt{3}x - y) - \sin\frac{k}{2}(\sqrt{3}x + y)\right] \tag{2.7}$$

and two solutions consisting of hexagonal cells

$$\xi = \sqrt{4/15}\,\varepsilon(k^2)\left[\cos kx + \cos\frac{k}{2}(\sqrt{3}y - x) + \cos\frac{k}{2}(\sqrt{3}y + x)\right] \tag{2.8}$$

$$\xi = \sqrt{4/15}\,\varepsilon(k^2)\left[\cos ky + \cos\frac{k}{2}(\sqrt{3}x - y) + \cos\frac{k}{2}(\sqrt{3}y + x)\right] \tag{2.9}$$

In the cases (2.6), (2.8) $\sqrt{3}k = mR_0^{-1}$, and in the cases (2.7), (2.9) $k = mR_0^{-1}$ $(m = 1, 2, \ldots)$. In both cases k^2 belongs to the interval (2.2) with $n = 0$.

It is important to note that (1.2) can be written in the gradient form

$$\xi_t = -\frac{\delta H\{\xi\}}{\delta\xi}, \quad H\{\xi\} = \int dx\, dy \left[\frac{\xi^2}{2} - \alpha(\nabla\xi)^2 + \frac{(\Delta\xi)^2}{2} + \frac{\xi^4}{4}\right] \tag{2.10}$$

This shows that the function H, like the energy in dissipative mechanical systems, does not increase with time, i.e.

$$\frac{dH}{dt} = \int \frac{\delta H}{\delta\xi}\xi_t\, dx\, dy = -\int \left(\frac{\delta H}{\delta\xi}\right)^2 dx\, dy$$

Therefore, of the solutions (2.4)-(2.9) the most stable solution will be the one which realizes the absolute minimum of the mean value of H per unit surface: $h = H/\int dxdy$. Corresponding computations show that the minimum value of $h = -{}^1/_6\varepsilon^4(k^2)$ is reached on the quasi one-dimensional structure.

We must confirm that the solutions (2.4) are also stable in the presence of small perturbations. Linearizing (1.2) with respect to small perturbations u on the background of the stationary solution (2.4), we obtain

$$u_t + u + 2\alpha\Delta u + \Delta^2 u + 2(1 + \cos(2\mathbf{kx}))\,u = 0, \quad \mathbf{k} = (k, 0) \tag{2.11}$$

The characteristic mode which may lead to instability, has the following general form:

$$\begin{aligned} u = \exp[\Omega(\mathbf{p})\,t]\,(a_+\cos\mathbf{p}_+\mathbf{x} + b_+\sin\mathbf{p}_+\mathbf{x} +{} \\ b_-\sin\mathbf{p}_-\mathbf{x} + a_-\cos\mathbf{p}_-\mathbf{x}), \quad \mathbf{p}_\pm = \mathbf{p} \pm \mathbf{k}, \quad |\mathbf{p}| \ll 1 \end{aligned} \tag{2.12}$$

The dispersion relation connecting Ω with $\mathbf{p}$ is found by substituting (2.12) into (2.11) and equating to zero the coefficients of the independent harmonics. The resulting fourth-order determinant is written in the form of a second-order determinant squared, and the equation becomes

$$[\Omega - \varepsilon^2(\mathbf{p}_+^2) + 2\varepsilon^2(k^2)]\,[\Omega - \varepsilon^2(\mathbf{p}_-^2) + 2\varepsilon^2(k^2)] - \varepsilon^4(k^2) = 0 \tag{2.13}$$

Equation (2.13) defines two branches of the increment $\Omega(\mathbf{p})$. The first branch is automatically negative for small p^2, since we have for it $\Omega(0) = -2\varepsilon^2(k^2)$, and the second branch vanishes as $p^2 \to 0$

$$\Omega(\mathbf{p}) = {}^1/_2\,(-\varepsilon^2(k^2) + 2(k^2 - 1)^2)(\mathbf{kp})^2 - \mathbf{p}^4 \tag{2.14}$$

(The first branch represents the root of the dispersion equation which is in fact extraneous, whose appearance is connected with the formal increase in order during the passage from the basic equation (2.11) to the truncated one /9/).

From (2.14) it follows that the quasi one-dimensional structure (2.4) is locally stable when the following condition, obtained in /8/ for the purely one-dimensional case, holds:

$${}^2/_3\lambda^2 \leqslant \varepsilon^2(k^2) \leqslant \lambda^2$$

i.e. in the region where the square of the amplitude of the fundamental (first) harmonic of the solution is equal to at least two thirds of its maximum value reached when $k^2 = 1$. Since the absolute minimum of h is reached in the solution (2.4) when $k^2 = 1$, it is in this structure, that the cylindrical front of evaporation appearing as a result of almost any initial conditions, must in the end undergo evolution. The results obtained concerning the stability of the solutions of (1.2) with cylindrical boundary conditions can be transferred, almost verbatim, to the case of an unbounded plane; the form of the solutions (2.4)-(2.9) remains practically unchanged.

We have a completely different situation in the case of an equation with quadratic non-linearity (1.1). In the unbounded plane the equation has, on the whole, no stable stationary solutions (at least of sufficiently simple form). On the cylindrical surface however, as will be shown below, stable solutions do exist. One of them is the quasi one-dimensional solution*

$$\xi(x) = 3\varepsilon(k^2)\sin kx - {}^9/_2\varepsilon^2(k^2) - {}^1/_2\varepsilon^2(k^2)\cos 2kx \tag{2.15}$$

(*Details of the study of such solutions on a cylindrical surface are given in the paper by: Malomed B.A. and Staroselskii I.E. Stability of quasiharmonic structures in gas flames. Preprint, Chernogolovka, 1983.)

The solution is stable, provided that the cylinder radius $R < 1 - \lambda/2$, since the characteristic modes of small perturbations, disturbing the structure (2.15), are forbidden by the condition of periodicity. When $R > 1 - \lambda/2$, the solution (2.15) becomes unstable with respect to e.g. the perturbation $u \sim \exp(\Omega t)\sin(yR^{-1})$.

When R finds itself, as a result of a further increase, with the narrow interval

$$1 - \lambda/2 \leqslant R \leqslant 1 + \lambda/2 \tag{2.16}$$

we first arrive at the value of k_y, permitted by the condition (2.1), for which the vector (k_x, k_y) falls in the region (2.2) $n = 1$. The solution which appears in this case in the form of rectangular cells

$$\xi = 3\varepsilon(k_x^2)\sin k_x x - {}^9/_2\varepsilon^2(k_x^2) - {}^1/_2\varepsilon^2(k_x^2)\times \cos 2k_x x + 3\varepsilon(k_y^2)\sin k_y y - {}^9/_2\varepsilon^2(k_y^2) - {}^1/_2\varepsilon^2(k_y^2)\cos 2k_y y \tag{2.17}$$

is locally unstable in the plane, but the characteristic modes of small perturbations which would destroy the structure (2.17) are forbidden in region (2.16) by the condition of periodicity. When R is increased further, the rectangular lattice will disappear since the solution no longer satisfies the boundary condition and a stochastic regime will obviously occur in this region. Equation (1.1) also admits of hexagonal solutions. These however are unstable even under the action of purely amplitude-type perturbations, i.e. perturbations which do not violate the symmetry of the solution. It follows that no boundary conditions can stabilize them.

Thus Eq.(1.2) has a family of stable cellular solutions on a cylindrical surface of any radius, and the most stable solution is the qusi one-dimensional one

$$\xi(x) = \sqrt{{}^4/_3}\lambda\cos x + O(\lambda^3)$$

When the cylindrical combustion front of a gaseous mixture described by (1.1) expands, a change must be observed at its face, from the quasi one-dimensional structure, to a rectangular parquet-like pattern, followed by the onset of an essentially non-stationary combustion mode. It should be noted that cellular structures were observed in gas flames experimentally /10/ and they definitely became stochastic for large values of R_0.

3. If the equations (1.1),(1.2) are specifed on a spherical surface of radius R, then the trivial solution $\xi\equiv 0$ in the weakly supercritical region $\lambda^2 \ll 1$ loses its stability with respect to small perturbations

$$\xi \sim e^{\Omega t}Y_{lm}(\theta,\varphi) = e^{\Omega t}P_{lm}(\cos\theta)\,e^{im\varphi}$$

provided that the following condition holds:

$$\varepsilon^2(l,R) = \lambda^2 - [l(l+1)/R^2 - 1]^2 \geqslant 0 \tag{3.1}$$

Here θ and φ are spherical coordinates, $Y_{lm}(\theta,\varphi)$ is a spherical harmonic and $P_{lm}(\cos\theta)$ is the associated Legendre polynomial. If

$$R \ll R_c,\quad R_c = (2\lambda)^{-1} \tag{3.2}$$

then narrow (of width λ^2) ranges of the values of the radius R exist for which an integer (and unique) value of l from (3.1) exists, and the distance separating these ranges is ~ 1.

When R becomes comparable with R_c, regions of stability of the unperturbed spherical front vanish. Henceforth, when $R \gg R_c$ expression (3.1) determines, like (2.2), the broad spectrum of characteristic modes which become excited for fixed R.

Here we shall only consider the case (3.2), since the opposite situation resembles the case discussed above. Then the dissipative structure will be described, provided that condition (3.1) holds, by the formula

$$\xi(\theta,\varphi) = a_{lm}P_{lm}(\cos\theta)\cos m\varphi + o(a_{lm}) \tag{3.3}$$

where the amplitude of the structure a_{lm} is of order $\varepsilon(l,R)$ or $\varepsilon^2(l,R)$ (see above). The value of l is found from (3.1), $0\leqslant m\leqslant l$. When $m=0$, these structures describe rings on the sphere, and are analogous to the quasi one-dimensional structures (2.4),(2.15), while when $m\geqslant 1$ they describe a lattice of spherical trapezia resembling the structures (2.5), (2.17). We shall calculate here the quantities a_{lm} for the equations (1.1),(1.2) for various special cases, and show the instability of all corresponding solutions.

To describe the solution of (1.1), it is sufficient to take into account the first term of the series whose beginning is given in (3.3), provided only that l is even and $m=0$. Then, substituting (3.3) into (1.1) we obtain

$$a_{l0} = \varepsilon^2(l,R)\frac{R^2}{2l+1}\times\left[\int_0^\pi\left(\frac{\partial P_{l0}(\cos\theta)}{\partial\theta}\right)^2 P_{l0}(\cos\theta)\sin\theta\,d\theta\right]^{-1} \tag{3.4}$$

(When l is odd, integration over $d\theta$ would make the right-hand side of (3.4) infinite). Since the amplitude obtained is of the second order of smallness, we can conclude at once that the structure is unstable.

Indeed, if we choose on the background of (3.3) a perturbation in the form $u\sim e^{\Omega t}Y_{lm_1}$ with any $m_1\neq 0$, then the stabilizing contribution towards the increment Ω due to the term $(\nabla\xi)^2$ will in every case be proportional to $\varepsilon^4(l,R)$, while the destabilizing contribution arising from the linear part of the equation will be equal to $\varepsilon^2(l,R)$.

In the case of odd l or $m\neq 0$, the amplitude of the stationary structure (3.3) is formed by all spherical harmonics Y_{LM} for which

$$\int_0^{2\pi}\int_0^{\pi} Y_{lm}(\nabla Y_{LM}\nabla Y_{lm})\sin\theta d\theta\, d\varphi \neq 0$$

The simplest solution of this type corresponding to $l=1,\ m=0$, should be sought in the form (there is no solution corresponding to $l=1,\ m=1$)

$$\xi = a_{10}\cos\theta + 3A\cos\theta\sin\theta\cos\varphi \tag{3.5}$$

Substituting (3.5) into (1.1) and equating to zero the coefficients accompanying the independent harmonics, we can find their amplitudes

$$a_{10} = \sqrt{5}\,\varepsilon(1, R), \quad A = (^{18}/_{11})\,\varepsilon^2(1, R) \tag{3.6}$$

Thus the amplitude of the fundamental (first) harmonic of the solution is proportional to $\varepsilon(1, R)$. The assertion also holds for any odd $l > 1$. Considering a small perturbation proportional to $e^{\Omega t}\sin\theta\cos\varphi$ on the background of the solution (3.5), (3.6), we can see that its instability increment is positive $\Omega = \varepsilon^2$. The stationary solutions will all odd $l > 1$ obviously possess this instability.

It should be noted that since the curvature is a stabilizing factor and the number l given by (3.1) increases as the curvature decreases, the instability of the structures (3.4), (3.5) and (3.6) corresponding to the smallest $l = 1, 2$ indicates at once the instability of all solutions of (1.1) on the sphere. The conclusion that no stationary stable structures are present in gas flames described by (1.1), agrees with the results of the numerical investigation carried out in /11/.

The fundamental harmonic of (3.3) is always sufficient to determine the stationary solution of (1.2), since the functions $P_{lm}(x)$ and $[P_{lm}(x)]^3$ have the same parity for all l and m. For example, the amplitude corresponding to $l=1$ and $m=0$ is found from the relation

$$-\varepsilon^2(1, R)\langle\cos^2\theta\rangle + a_{10}^2\langle\cos^4\theta\rangle = 0$$

$$\left(\langle f(\theta)\rangle \equiv \int_0^{\pi} f(\theta)\sin\theta\, d\theta\right)$$

and from these we have

$$a_{10} = \sqrt{^5/_3}\,\varepsilon(1, R) \tag{3.7}$$

The unique characteristic small-perturbation mode which may lead to instability of this solution, has the following form in the zero order in ε:

$$u = e^{\Omega t}\sin\theta\cos\varphi \tag{3.8}$$

We can show, however, that the use of (3.8) leads only to a reduction in the destabilizing contribution $\varepsilon^2(1, R)$ to the increment in Ω, which should therefore be sought with an accuracy to terms of order $\varepsilon^4(1, R)$. To do this, we must find a correction to the characteristic mode (3.8) proportional to ε^2

$$u_1 = -{}^1/_{49}\varepsilon^2(1, R)\,P_{31}(\cos\theta)\cos\varphi \tag{3.9}$$

Substitution of (3.6)-(3.9) into (1.2) shows that the solution found from the equations (3.3), (3.6) is unstable, and $\Omega = (6/343)\,\varepsilon^4(1, R)$. The amplitude of the solution proportional to Y_{11} and the increment of its instability are both identical with those just found. We also give the value $a_{20} = (56/339)\,\varepsilon$ for the solution proportional to Y_{20}. This solution is also unstable, with an increment proportional to $\varepsilon^4(2, R)$. We must however remember that introducing small terms into (1.2), such as ξ^5, will alter the increment by an amount of order ε^4, and this can, in principle, change its sign. Therefore the solution (3.7) and another similar solution (see below) should be regarded, within the accuracy with which (1.2) describes the physical problem, as neutrally stable.

We can also obtain compact expressions for the amplitudes of cellular structures, and study their stability in the case when the radius of the sphere is $R \gg 1$ (still remaining within the region (3.2)). Then the number l determined by (3.1) is also large, and the asymptotic expressions for the Legendre polynomials can then be used /12/. All amplitudes are conveniently determined not by using relations (3.3), but from

$$\begin{aligned}\xi &= A_{lm}\sin\left[\left(l+\frac{1}{2}\right)\theta+\frac{\pi}{4}\right]\sin^{-1/2}\theta\cos m\varphi, \quad m \text{ even}\\ \xi &= A_{lm}\cos\left[\left(l+\frac{1}{2}\right)\theta+\frac{\pi}{4}\right]\sin^{-1/2}\theta\cos m\varphi, \quad m \text{ odd}\end{aligned} \tag{3.10}$$

Substituting (3.10) into (1.2) we obtain

$$A_{l0}=2\sqrt{\frac{\pi}{3}}I_l^{-1/4}\varepsilon(l,\ R) \tag{3.11}$$

for the quasi one-dimensional structure, and

$$A_{lm}=\frac{4}{3}\sqrt{\pi}I_l^{-1/4}\varepsilon(l,\ R) \tag{3.12}$$

for all parquet-like structures outside the dependence on m, where I_l is a quantity equal to $2\ln l$ with logarithmic accuracy. (Since $l\gg 1$, we replace in (3.11), (3.12) the quantity $l+{}^1/_2$ by l). We note that when the radius R increases, the curvature of the surface connected with formation of the dissipative structure is small

$$A_{lm}/R\sim(R\sqrt{\ln R})^{-1}\to 0 \quad \text{as} \quad R\to\infty$$

To show that all parquet-like structures are unstable, it is sufficient to take, on their background, a small perturbation independent of φ

$$u\sim e^{\Omega t}\sin\left[\left(l+\frac{1}{2}\right)\theta+\frac{\pi}{4}\right]\sin^{-1/2}\theta$$

which yields $\Omega={}^2/_3\varepsilon^2(l,R)$. It can also be shown that the quasi one-dimensional structure is stable to perturbations proportional to Y_{lm}, with even m. When the perturbation is proportional to $Y_{l1}(\theta,\varphi)$, the stabilizing contribution towards the increment is found to be equal to ${}^3/_4A_{l0}{}^2(I_l/\pi)$, and this shortens, as far as the accuracy is concerned, the destabilizing term $\varepsilon^2(l,R)$ irrespective of the method used to compute the quantity I_l. Therefore, to explain the stability of the ring structures we must, as was shown above, consider the correction to the characteristic mode proportional to ε^2. As a result we find, that all ring structures with an increment of the order of $\varepsilon^4(l,R)$ are unstable.

Note that we could seek the solutions of (1.1), (1.2) on the spherical surface proportional, to a first approximation, to the combination of harmonics Y_{lm} with different m and equal l, incident on the region (3.1). We find however, that, at least when $l=1,2,3$ there are no such solutions, neither in the case of (1.1), nor of (1.2). Equations containing both quadratic and cubic non-linearities, may have such solutions.

4. Let us consider some one-dimensional solutions of (1.1), (1.2) in a plane, the solutions themselves possessing non-trivial topological properties /13/

$$\xi(x)=a(x)\sin kx+o(a),\quad |\nabla a|\ll 1 \tag{4.1}$$

Substituting (4.1), into e.g. (1.2) we find, that such a solution exists only when $k^2=\alpha$ and the amplitude $a(x)$ has the form

$$a(x)=\sqrt{{}^4/_3}\,\lambda\,\mathrm{th}\left(\frac{\lambda}{2\sqrt{2}}x\right) \tag{4.2}$$

Its "topological charge" /13/ manifests itself in the fact that the amplitude changes its sign on changing from $x=-\infty$ to $x=+\infty$. In other words, the phase changes, at the point $x=0$ where the amplitude becomes zero, by π in a discontinuous manner. The solution (4.1), (4.2) is stable and the analogous solution of (1.1) is unstable on an unbounded plane.

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REFERENCES

1. LANDAU L.D. and LIFSHITS E.M., Mechanics of Continuous Media. Moscow, Gostekhizdat, 1954.
2. GOR'KOV L.P., Stationary convection in a plane fluid layer near the critical mode of heat transfer. Zh. eksperim. i teor. fiz. Vol.33, No.2, 1957.
3. ISTRATOV A.G. and LIBROVICH V.B., Stability of flames. In book: Results in Science. Hydromechanics. Moscow, VINITI, 1966.
4. ALDUSHIN A.P., KASPARYAN S.G. and SHKADINSKII K.G., Formation of a two-dimensional cellular structure in a thermodiffusive flame. Dokl. Akad. Nauk SSSR, Vol.247, No.5, 1979.
5. ANISIMOV S.I., TRIBEL'SKII M.I. and EPEL'BAUM YA.G. Instability of the plane evaporation front when laser radiation interacts with a material. Zh. eksperim. i teor. fiz., Vol.57, 1981.
6. SCOTT A.C., The Electrophysics of a Nerve Fiber. Revs Mod. Phys., Vol.47, No.2, 1975.
7. SIVASHINSKY G.I., On Flame Propagation under Conditions of Stoichiometry. SIAM J. Appl. Math., Vol.39, No.1, 1980.
8. ANISIMOV S.I., GOL'BERG S.M., MALOMED B.A. and TRIBEL'SKII M.I., Two-dimensional weakly supercritical structures in laser sublimation waves. Dokl. Akad. Nauk SSSR, Vol.262, No.5, 1982.
9. HAKEN G., Synergetics. Moscow, Mir, 1980.

10. Non-steady Flame Propagation. Oxford: Pergamon Press, 1966.
11. MICHELSON D.M., SIVASHINSKY G.I., Thermal Expansion Induced Cellular Flames. Combust. and Flame, Vol.48, No.2, 1982.
12. BATEMAN H. and ERDELYI A., Higher Transcendental Functions. Vol.2, McGraw-Hill, New York, 1953.
13. POLYAKOV A.M., Isomeric States of Quantum Fields. Zh. eksperim. i teor. fiz., Vol.68, No.6, 1975.

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THE THERMAL PROBLEM FOR A SUBMERGED STREAM*

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The general solution of the thermal problem of convective heat conduction with volume heat dissipation caused by viscous dissipation of kinetic energy of the fluid, whose velocity field is determined by the exact solution /1/ of the Navier-Stokes equations, is considered for a submerged stream. The possible formulation of the heat problem and the characteristic behaviour of the solutions are investigated. The solutions obtained have a special feature, namely the existence, under specified conditions, of two regimes of convective heat exchange.

A particular solution of the problem in question corresponding to a point heat source superimposed on the steam source was obtained in /2/, without taking into account the dissipative heat emission. The solution corresponds to the first term of the expansion of the temperature in a series in multipoles

$$T(R,\ \theta)=\sum_{n=1}^{\infty}\tau_n(\theta)\,R^{-\alpha_n} \tag{1}$$

where R, θ are spherical coordinates and the angle θ is measured from the stream axis. The appearance in (1) of the fractional indices α_n is connected with the presence, in the equation of heat conduction

$$LT=\frac{\nu}{2c_p}\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right)^2,\quad L=(u,\ \nabla)-a\Delta \tag{2}$$

of the convective term which changes the spectrum of the operator L.

The velocity field for a submerged stream has the form /1/

$$u_R=\nu\frac{y'(x)}{R},\quad u_\theta=\nu\frac{y(x)}{\sqrt{1-x^2}}\frac{1}{R};\quad y(x)=-2\frac{1-x^2}{A-x},\qquad x=\cos\theta \tag{3}$$

The coefficients of kinematic viscosity ν and thermal conductivity a are assumed to be constant, $A>1$ is a constant connected monotonically with the momentum of the stream

$$I=16\pi\rho\nu^2A\left[1+\frac{4}{3(A^2-1)}-\frac{A}{2}\ln\frac{A+1}{A-1}\right] \tag{4}$$

and according to (4) $I\to 0$ as $A\to\infty$ and $I\to\infty$ as $A\to 1$.

The expansion (1) holds only for the solution of the homogeneous equation (2). The solution of the inhomogeneous equation of heat conduction (2) contains, apart from (1), a term whose form is determined by the dissipative heat source. Without the convective terms in the homogeneous equation the expansion (1) assumes the classical form, with $\alpha_n=n$ and τ_n being spherical functions. In the general case $\alpha_n\neq n,\ n>1$. According to (3), the dissipative function in (2) is proportional to R^{-4}, and this generates in (1), with one exception which will be noted below, an additional term of the form $z(x)R^{-2}$ corresponding to the particular solution of the inhomogeneous equation.

Substituting (1) and (3) into (2), we obtain for $\alpha_n\neq 2$

$$(1-x^2)\tau_n''-2x\tau_n'+\Pr(y\tau_n'+\alpha_ny'\tau_n)+\alpha_n(\alpha_n-1)\tau_n=0 \tag{5}$$

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